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Optimal Entry and the Marginal Contribution of a Player

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Abstract

We introduce the concept of "the marginal contribution of a player(firm)" and use it to derive conditions for optimal entry in various industrial situations. It turns out that, in a competitive economy with a finite number of goods but with a continuum of potential firms, the marginal contribution of a firm coincides with the profit of the firm, and so the optimal condition for entry is that the marginal firms should receive zero profit. We also study the marginal contribution in the monopolistic competition markets and establish the "excess entry theorem" in a new setting.

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1 Introduction

The optimal[1] number of firms in an industry could be either one, two or many depending on the market structure. The main purpose of this paper is to introduce the concept of "the marginal contribution of a player(a firm in the sequel)" and use it to derive conditions for optimal entry in various industrial situations. We consider an economy with a finite number of goods but with a continuum of potential firms. The "marginal contribution of a firm" is defined roughly as (the limit, as the measure of the firm approaches zero, of) the difference between the maximal welfare that the economy can attain with the firm and without it. It turns out that, when there are fixed costs, not all firms should produce positive outputs even if they have the same production technology. Under perfect competition, the marginal contribution of a firm coincides with the profit of the firm, and so the optimal condition for entry is that the marginal firms should receive zero profit.

Our concept of the "marginal contribution of a firm" is closely related to the idea which welfare economists, e.g., Kahn (1935) and Hicks (1939), had in mind in discussing optimal industrial structure or the "total conditions" for optimality. The game theoretic concept of Shapley value (see, e.g., Shapley (1953), Aumann and Shapley (1971)) is also related to the present concept. But whereas the Shapley value is the "expected pay off" of the game when all agents are arranged in random order, in our definition, firms are ordered according to their productivity where productivity is defined in a natural way. Using this concept we derive conditions for optimal entry which were obtained verbally or in a partial equilibrium framework by Kahn (1935), Hicks (1939) and obtained in a general equilibrium framework by Negishi (1962,1972)).[2] See, also Makowski (1980) and Ostroy (1980) for related discussions.

Our analysis stands in contrast with previous studies in that the set of agents are contained in a non-atomic measure space. The same approach is, also useful in analyzing the problems of the monopolistic competition market, as we will show in Section 4. We establish a version of excess entry theorem which conveys a similar message as in Suzumura and Kiyono (1987) established for the oligopolistic market. This approach, which follows the

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[1]Our criterion of optimality here is the maximality of the Bergson-Samuelson type social welfare function. We assume away the problems associated with imperfect information and suppose that the government can attain the optimum by some policy means.

[2]Negishi's theorems state that (i) if it is known that positive profit is impossible for the new firm under prices ruling before entry, entry should not be made and that (ii)if the new firm is running without a loss after entry, then the firm should have entered after all (see Negishi (1972)). The last statement needs a careful interpretation if the incumbent firms are not the most desirable from the welfare viewpoint.

procedure of Aumann (1964,1975) has the advantage that the marginal contribution of a firm can unambiguously be expressed in terms of the prices and the allocation of the economy, and the convexity assumptions on preferences and technologies can be relaxed to a certain extent.

2 A Preliminary example

In order to clarify the nature of the problem and motivate the analysis in the following sections, we first present a simple example and derive optimal conditions for entry in this case. In this section all firms are treated discretely, and the analysis is informal for reasons that will be explained below.

Suppose that the welfare of an economy can be expressed by the utility function

$$u = x \cdot (a - l) \tag{1}$$

of a representative consumer, where x is the amount of the consumption good available to him, l is the amount of labor he supplies and a is a positive number representing the maximal amount of labor that he can supply in a fixed time (thus $a - l$ represents consumption of leisure). Let $\underline{J} = \{1, 2, 3, ...\}$ denote the set of firms in the economy that can potentially produce the consumption good, and assume that the production function of the j -th firm ($j \in \underline{J}$) can be written as

$$x_j = \begin{cases} \sqrt{l_j - b_j} & \text{if } \; l_j > b_j \\ 0 & \text{if } \; l_j \leq b_j \end{cases} \tag{2}$$

where l_j the amount of labor, x_j is the amount of production and b_j is a given non-negative number representing the fixed input of the j-th firm.Let us first consider the situation where only firms in a subset J of $\underline{J}$ are active. (This means that $l_j = 0$ for all $j \in \underline{J} \backslash J$). If some firms in J are not producing positive outputs, then the consumer need not supply positive amount of labor to these firms. Hence in considering the social optimum we may assume that all of the members of J are producing positive outputs. We now formulate the problem (P_J) for each such $J \subset \underline{J}$ as :

(P_J) Maximize

$$u = x \cdot (a - l)$$

subject to

$$x = \sum_{j \in J} \sqrt{l_j - b_j} \tag{3}$$

and

$$l = \sum_{j \in J} l_j. \tag{4}$$

From this we easily obtain the familiar marginal conditions for optimality
:

$$\frac{a-l}{x} = 2\sqrt{l_j - b_j} \tag{5}$$

Hence, in view of (1), (3) and (4), we have the following optimal production allocation

$$x_j^*(J) = \sqrt{l_j - b_j} \quad (j \in J). \tag{6}$$

$$l_j^*(J) = (a - \sum b_i)/3n + b_i \quad (j \in J) \tag{7}$$

and the corresponding optimal utility

$$u^*(J) = 2\sqrt{n}(a - \sum b_j)^{\frac{3}{2}}/3\sqrt{3}, \tag{8}$$

where the summations are over J , and n is the number of firms producing positive outputs, i.e., the cardinality of J . (The above results show that J must be chosen so that $a - \sum b_j > 0$).

In the next step we allow J to vary, and choose $x_j^*(J)$ and $l_j^*(J)$ to maximize $u^*(J)$. To simplify the analysis we shall suppose that the firms are arranged so that

$$\text{if} \quad j < k \quad \text{then} \quad b_j \leq b_k \tag{9}$$

This implies that the production function of the j-th firm is uniformly above that of the k-th firm for $k > j$. Thus, if the k-th firm is producing positive outputs at the social optimum, then so should the j-th firm, for any $j < k$. Hence in order to choose the optimal set of firms, J, it is enough to determine the optimal number, n, of firms that will produce positive output.

In the characteristic function form game (u^*, J) with the characteristic function u^* and the player set J , the $marginalworthofaplayerj$ to coalition $S (S \subset J)$ is defined by

$$u^*(S \cup \{j\}) - u^*(S) \quad \text{for } j \notin S$$

Hence writing $u^*[n]$ for $u^*(J)$ (where n is the cardinality of J) it may seem natural to define the marginal worth of the n-th firm by $u^*[n] - u^*[n-1]$ or, supposing that $u^*[n]$ is defined for all real numbers, by du^*/dn . Actually, it turns out to be more convenient to define it by

$$-\frac{du^*/dn}{\partial u^*/\partial l} \tag{10}$$

which is also independent of the choice of utility functions. (The denominator represents the marginal disutility of labor evaluated at the optimum allocation.)This corresponds to what we will later call the *marginal contribution* of the firm.

In the special case where $b_j = b$ for all j , we have

$$u^*[n] = 2\sqrt{n}(a - nb)^{\frac{3}{2}}/3\sqrt{3}. \tag{8'}$$

Hence if we allow n to take on all positive values, we have

$$\frac{du^*}{dn} = \frac{\sqrt{a - nb}(a - 4nb)}{3\sqrt{3n}} \tag{11}$$

Since, by (5), all $l_j^{*'}$ s are equal in this case, in view of (1),(3),(4) and (7), we obtain

$$-\frac{\partial u^*}{\partial l} = \sqrt{a - nb}\sqrt{n}/\sqrt{3} \tag{12}$$

and

$$\frac{\partial u^*}{\partial x} = 2(a - nb)/3. \tag{13}$$

Equations (11) and (12) then imply

$$-\frac{\partial u^*/dn}{\partial u^*/\partial l} = \frac{a - 4nb}{3n}. \tag{14}$$

Now if the price vector $(-u_x^*/u_l^*, 1)$ is used to evaluate the profit π of the firm, we have, from (6), (7), (12) and(13)

$$\begin{aligned} \pi &= -\frac{u_x^*}{u_l^*}x_j^* - l_j^* \\ &= \frac{a - 4nb}{3n} \end{aligned} \tag{15}$$

Comparing (14) with (15) we may conclude that *the marginal contribution of the firm is the profit of the firm.*

The present analysis, which dealt with the case of a finite number of firms, is somewhat informal because the marginal contribution was not defined accurately. In the following sections, we shall rigorously establish similar results in more general settings without restricting ourselves to the special production functions and the utility function of this model.

Remarks (a) In the special case where $b_j = b > 0$ for all j, (11) shows that the optimal number of the firms in the industry is given by $a/4b$, if it is an integer. This implies that not all firms should stay in the industry even if they have the same technology.

(b) If, moreover, $b_j = 0$ for all j, then $u^*[n]$ is an increasing function of n, and there is no optimal number of firms for the economy.

(c) That the marginal worth is an increasing function with respect to the coalition size is a characteristic feature of the convex game which has been studied by Shapley(1971),Ichiishi(1981) and Topkis(1987), among others. Remark(a) shows that the present model contains an example of a non-convex game.

(d) As a model of entry in a free market, the discrete model must rely seriously on the assumption that entry occurs in the order of superiority in technology as expressed in (9). It is easy to construct an example in which (i) a finite number of firms are making positive profits and that (ii) a technologically superior firm incurs a loss should it enter the market. To see this, slightly increase the parameter b_i of an incumbent firm in the model of Remark(a).

3 The Marginal Contribution and the Efficiency Price

In this section we consider two different models of an economy with a finite number of goods but with a continuum of firms. There are no restrictions on prices or quantities of the goods traded and monopolies are ruled out. In both of these models firms are assumed to be arranged in a certain natural order, and we consider the overall effects of "another" firm joining an industry. The performance of the economy is considered to be expressed by a real valued function, which we may call an objective function or a welfare function. *The marginal contribution of a firm* is then defined as the limit, as the measure of the firm approaches zero, of the increase in maximum welfare, divided by the marginal contribution to welfare (marginal utility) of a numeraire, say labor. (see, also the discussion below). Equation(10) is the expression for this in the economic model of Section 2. For the definitions of economic concepts not defined here we refer to Samuelson (1947), Debreu (1959) and Arrow-Hahn (1971). The main result that we establish in this section is:

Theorem 1

In the classical[3] *Arrow-Debreu competitive economy, the marginal contribution of a firm is equal to the profit of the firm in terms of the efficiency prices.*

An efficiency price vector in terms of the numeraire good is the vector of marginal rates of substitution when they exist. In general it is defined by the normal vector of the separating a hyper plane to, say, the production set . Since competitive prices are also efficiency prices (cf. Debreu (1957) or Arrow-Hahn (1971)), Theorem 1 implies:

Theorem 2

Under the same assumption as in Theorem 1, the optimal condition for the entry of firms is that the profit of the marginal firm should equal to zero.

We will prove Theorem 1 under two slightly different sets of assumptions in models A and B. Model A is a continuum analogue, extended in several respects, of the example in section 2. It is assumed that the welfare of the economy is described by the utility function of a representative consumer. Model B is quite general in its treatment of production technology, but consumers' demands for goods are assumed to be given exogenously.

Model A

Let us assume that the utility function of a representative consumer is given by

$$u = (x_1, x_2, a - l) \tag{16}$$

where $x_i (i = 1, 2)$ denotes his consumption of good i, l is his labor supply and a is a given positive number representing the maximum amount of labor that he can supply. We make

Assumption A.1

$u(\cdot)$ is increasing, strictly quasi-concave and twice continuously differentiable. The set of firms that can potentially be in industry i is represented by a bounded interval $\underline{T_i} (i = 1, 2)$. We suppose that $\underline{T_1}$ and $\underline{T_2}$ are disjoint.

For each $i (i = 1, 2)$, let the production function of firm t be denoted by

$$x_i(t) = \begin{cases} f_i(l_i(t) - b_i(t), t) & \text{if} \quad l_i(t) > b_i(t) \\ 0 & \text{if} \quad l_i(t) \leq b_i(t), \end{cases} \quad (i = 1, 2) \tag{17}$$

[3] This usually means the economic environment with convex preferences and convex production technologies and with no externalities. However the term "classical" is used here in a somewhat broader sense than usual. Firms may require to use a fixed amounts of input when they produce positive amount of outputs although no inputs are required when no outputs are produced. Hence the average cost curve is decreasing when output levels are small.

where $x_i(t)$ is the density of production of good i and $l_i(t)$ (of which $b_i(t) > 0$ is a fixed amount) is the density of labor input for firm t in industry i. This means that given $l_i(t)dt$ of labour the firm can produce $f_i(l_i(t) - b_i(t), t)dt$ of the product if $l_i(t) > b_i(t)$. (See, e.g., Aumann (1975) or Aumann-Shapley (1971) for a related way of representing agents.) We make

Assumption A.2

For each t $f_i(\cdot, t)$ is increasing, strictly concave[4] and twice continuously differentiable. For each x, $f_i(x, \cdot)$ is continuous except possibly at a finite number of points, $(i = 1, 2)$.

The present model can be generalized to the case of any finite number of goods. Model B allows the existence of intermediate goods. Let us denote by $T_i \in \underline{T_i}$ the set of firms actually producing positive outputs in industry $i (i = 1, 2)$. To simplify the analysis we make

Assumption A.3

T_i is a disjoint union of a finite number of intervals T_{ik} $(k = 1, ..., i_k)$in$\underline{T_i}$.

We may suppose (as was explained in section 2) that all firms in T_i are actually producing positive outputs. In the sequel we shall often write e.g.,$\int_T f$ instead of $\int_T f dt$. The demands for goods are satisfied if

$$x_i \leq \int_{T_i} x_i(t) \quad (i = 1, 2) \tag{18}$$

and

$$l \geq \int_{T_1} l_1(t) + \int_{T_2} l_2(t). \tag{19}$$

Since u$(\cdot)$ is increasing, when finding the optimum, we may replace the inequalities in (18) and (19) by equalities. And if we extend the definitions of $l_i(t)$ and $x_i(t)$, by setting them equal to zero outside T_i, we may replace the domain of integration, T_i, by $T = T_1 \cup T_2$. Thus for each of T_1 and T_2, we formulate the problem (P_T) as:

(P_T) Maximize

$$u = u(\int_T x_1(t), \int_T x_2(t), a - \int_T (l_1(t) + l_2(t))) \tag{20}$$

subject to

$$x_i(t) = f_i(l_i(t) - b_i(t), t) \quad (i = 1, 2) \tag{21}$$

The existence of the maximum and some other related properties will be discussed in Section 5 in a more general framework, in which we will assume that $x_i(t)$ and $l_i(t)$ are Borel measurable functions. For the present

[4] We will argue below that this assumption is not practically important as is the case in the discrete economy.

we assume that the maximum exists and impose the following conditions on admissible functions:

Assumption A.4

$l_i(t)$ and $x_i(t)$ are continuously differentiable in the interior of each of the sub-intervals T_{ik} as defined in (A.3).

The problem(P_T) can easily be solved by substituting (21) into (20). Taking the variational derivative (see, e.g., Gelfand and Formin ((1963) pp.27-28) of u with respect to l_i, we know that the conditions for the extremum are

$$\frac{\partial u}{\partial x_i}\frac{\partial f_i}{\partial l_i} + \frac{\partial u}{\partial l} = 0 \quad (i = 1, 2) \tag{22}$$

These are nothing but the familiar marginal conditions for optimality. We next let $T_i(i = 1, 2)$ vary and consider the effects of the change on the optimal solutions of (P_T). To simplify the analysis we make

Assumption A.5

The left end-point of each sub-interval of $T_i(i = 1, 2)$, as defined in (A.3), and the number of these sub-intervals, are known.

The left end point represents (technologically) the most superior firm in the industry. We may suppose that (A.5) is satisfied if there are only a finite number of potential types of firms in an industry. More generally, (A.5) is satisfied if it is possible to classify firms into a finite number of groups in such a way that, within each of the groups, the production function of one firm is uniformly above or below that of another. Owing to (A.5) we need only consider changes in the right end-points of the sub-intervals representing the most inferior firm. Let us consider the effects of a change in a right end point, $\alpha = t_{is}$, of T_{is}.

Differentiating $u(\cdot)$, along the optimal path, with respect to α, we have (denoting by j the index different from i)

$$\begin{aligned} \frac{du}{d\alpha} &= \frac{\partial u}{\partial x_i}(x_i(\alpha) + \int_T \frac{\partial f_i}{\partial l_i}\frac{dl_i}{d\alpha}) + \frac{\partial u}{\partial x_j}(\int_T \frac{\partial f_j}{\partial l_j}\frac{dl_j}{d\alpha}) \\ &\quad + \frac{\partial u}{\partial l}(l_i(\alpha) + \int_T \frac{d(l_1 + l_2)}{d\alpha} \end{aligned} \tag{23}$$

Noticing that $\partial u/\partial x_i$ and $\partial u/\partial l$ are independent of t, we have from (22), and (23),

$$-\frac{\partial u}{\partial \alpha}/\frac{\partial u}{\partial l} = -\frac{\partial u}{\partial x_i}/\frac{\partial u}{\partial l} \times x_i(\alpha) - l_i(\alpha) \tag{24}$$

This means that the $marginal contribution of firm \alpha$ (the left hand side) is equal to the profit of the firm in terms of the efficiency price vector,

$$p = (-\frac{\partial u}{\partial x_i} / \frac{\partial u}{\partial l}, 1) \tag{25}$$

(the right hand side).

Model B

Let n be the number of goods in the economy and, for each $i \in N = \{1, 2, ..., n\}$,let $\underline{T_i}$ be a bounded interval in the real line R. We consider $\underline{T_i}$ to be the set of all potential firms in industry i. We assume that $\underline{T_i}$ and $\underline{T_j}$ are disjoint for $i \neq j$ .If good i is not the product of any firm, we take $\underline{T_j}$ to be empty. For each $i, j \in N$ with $i \neq j$, and each $t \in \underline{T_i}$ let $y_j^i(t)$ be the density of good j used (of which $b_j^i(t)$ is a fixed amount) in the production of good i by firm t, and let $y_i(t)$ be the density of its output. For simplicity we assume that there are no joint outputs and we write the firms' production functions as

$$y_i(t) = \begin{cases} f^i(\tilde{y}^i(t) - \tilde{b}^i(t), t) & \text{for} \quad \tilde{y}^i(t) \geq \tilde{b}^i(t) \\ 0 & \text{otherwise} \quad (i \in N, t \in \underline{T_i}) \end{cases}, \tag{26}$$

where $\tilde{y}_i(t) = (y_1^i(t), y_{i-1}^i(t), y_{i+1}^i(t), ..., y_n^i(t)$ and similarly for $\tilde{b}^i(t)$. We make the following assumptions on the production technology:

Assumption B.1

For each t, $f^i(\cdot, t)$ is increasing and twice continuously differentiable, and for each $y^i, f^i(y^i, \cdot)$ is continuous except perhaps at a finite number of points.

Assumptions must also be made on the asymptotic behavior of $f^i(\cdot, t)$, in order to guarantee the existence of a maximum of the problem to be formulated below. This point will be discussed in the Appendix so, for the moment, we will not worry about the problem of existence. Let $T_i \subset \underline{T_i}$ denote the set of firms producing positive outputs in industry $i (i \in N)$ and impose:

Assumption B.2

The same as assumption A.3 in model A.

Letting c_i $(i = 1, 2, ..., n-1)$ denote the aggregate net demand for good i ; that demand will be satisfied if

$$c_i \leq \int_{Ti} y_i(t) dt - \sum_{j \neq i} \int_{Tj} y_i^j(t) dt \tag{27}$$

Extending the definitions of $y_j^i(t), y_i(t)$ and $b_j^i(t)$, by defining them to be equal to zero outside T_i, we may replace T_i in (27) by $T = \cup T_i$. Now, for a given (T_i) $(i \in N)$, we formulate the problem (P_T) as

(P_T) Maximize

$$\int_T f^n(\tilde{y}^n(t) - \tilde{b}^n(t), t)dt - \sum_{j \neq n} \int_T y_n^j(t)dt \tag{28}$$

subject to

$$c_i = \int_T f^i(\tilde{y}^i(t) - \tilde{b}^i(t), t)dt - \sum_{j \neq i} \int_T y_i^j(t)dt$$
$$(i = 1, ..., n-1) \tag{29}$$

where $c_i (i = 1, ..., n-1)$ and $\tilde{b}^i(t)(i = 1, 2, ..., n)$ are assumed to be given. A natural interpretation of the problem is that it is to minimize the sum of the labor inputs of the economy on the conditon that specified demands are satisfied. The equality in (29) is due to the assumption that $f^i(\cdot, t)$ is increasing.

As in the previous model, the following conditions are imposed on the admissible functions:

Assumption B.3

All $y_j^i(t)$ and $y_i(t)$ are continuously differentiable in the interior of each sub-interval, T_{ik}, as defined in (A.3).

Following the standard procedure (cf. Gleaned and Forman ((1963) pp.43-46) we write the Lagrangean of the problem as

$$\sum_{i=1}^{n} p_i (\int_T (f^i(\tilde{y}^i(t) - \tilde{b}^i(t), t) - \sum_{j \neq i} y_i^j(t))dt - c_i), \tag{30}$$

with $p_n = 1$ and $c_n = 0$, and obtain the Euler conditions for optimality:

$$p_i \frac{\partial f^i(t)}{\partial y_j^i} = p_j \quad (i, j \in N \;\; t \in T_i) \tag{31}$$

where we set $f^i(t) = f^i(\tilde{y}^i(t) - \tilde{b}^i(t), t)$. These are the familiar marginal conditions for optimality. We next let (T_i) (i $\in$ N) vary and consider the effects of the change on the optimal solutions of (P_T). To simplify the analysis we impose

Assumption B.4

The same as (A.5) in Model A.

With this assumption, we need only consider changes in the right end points of the sub-intervals T_{ks}.

Differentiating (29) with respect to a right end point $\alpha = t_{ks}$, we have (noticing that the optimal solutions of y_i^j, y_i and p_i are functions of α)

$$m(\alpha) = p_k y_k(\alpha) - \sum_{j \neq k} p_i y_i^k(\alpha) + \int_T \sum_{i \in N} p_i (\sum_{j \neq i} \frac{\partial f^i(t)}{\partial y_j^i} \frac{\partial y_j^i}{\partial \alpha} - \sum_{j \neq i} \frac{\partial y_i^j}{\partial \alpha})dt \tag{32}$$

But the terms in parenthesis cancel out since, by (31),

$$\begin{aligned}\sum_{i\in N} p_i \sum_{j\neq i} \frac{\partial f^i(t)}{\partial y^i_j}\frac{\partial y^i_j}{\partial \alpha} &= \sum_{i\in N}\sum_{j\neq i} p_i \frac{\partial f^i(t)}{\partial y^i_j}\frac{\partial y^i_j}{\partial \alpha} \\ &= \sum_{i\in N}\sum_{j\neq i} p_j \frac{\partial y^i_j}{\partial \alpha} \\ &= \sum_{i\in N}\sum_{j\neq i} p_i \frac{\partial y^j_i}{\partial \alpha}. \end{aligned} \tag{33}$$

The last two relations imply

$$m(\alpha) = p_k y_k(\alpha) - \sum_{j\neq k} p_i y^k_i(\alpha). \tag{34}$$

Since the marginal contribution of y_n to the objective function is $p_n = 1$, we know that $m(\alpha)$ is the marginal contribution of firm $\alpha = t_{ks}$. By (34), it is equal to the profit of the firm in terms of the efficiency price vector $(p_1, p_2, ..., p_{n-1}, 1)$.

4 The Marginal Contribution in a Monopolistic Model

In this section we apply the previous analysis to derive the marginal contribution of a firm in a simple model of monopolistic competition. The contribution to welfare of a monopolistically competitive firm is calculated under the assumption that the behavior rule of the other firms in the markets are unaltered. Another possible interpretation of the model will be discussed below.

Model C.

The basic framework of the model is the same as that of model A except that there is only one industry in the present case. Using the previous notation let

$$U = u(x) + a - l \tag{35}$$

be the utility function of the representative consumer which is the sum of utility from a consumption good $u(x)$ and the leisure $a - l$. We assume that u is an increasing concave function. The industry has a continuum of potential firms which we denote by $\underline{T}$. We assume that $\underline{T}$ is a bounded interval in the real line R . The production function of firm t of the industry is denoted by

$$x(t) = \begin{cases} f(l(t) - b(t), t) & \text{if} \quad l(t) > b(t) \\ 0 & \text{if} \quad l(t) \leq b(t) \end{cases} \tag{36}$$

We will assume that f is concave in the region $l(t) > b(t)$ We make Assumption (A.1) and Assumption (A.2) of Section 3 applied for the single industry case.

The profit of the industry is expressed as

$$\pi = \int_T (px(t) - l(t))dt \tag{37}$$

where p is the price of the product in terms of the wage rate. We denote the (inverse) demand function of the consumer as

$$p = P(x), \quad P(x) = u'(x) \tag{38}$$

We will assume that

Assumption C.1

$$P'(x) > 0,$$
$$P(x) + xP'(x) > 0$$
$$2P'(x) + xP"(x) > 0$$

The first inequality means that the marginal utility of the good is positive. The second and the third inequalities say that the marginal revenue is positive and decreasing. It is also assumed that monopolistic firms in the industry maximize their joint profits π with respect to $l(\cdot), x(\cdot)$, and T , knowing the consumer's demand function for their good P . See, Remark (a) below for another interpretation. We make assumptions (A.4) and (A.5).

We now consider the problem:

(P) Maximize

$$\pi = \int_T [P(x)x(t) - l(t)]dt \tag{39}$$

subject to

$$\begin{aligned} x &= \int_T x(t)dt \\ &= \int_T f(l(t) - b(t), t)dt \end{aligned} \tag{40}$$

First we solve the problem considering that x and T are fixed. We set the Lagrangean of the problem as

$$L = \int_T [Pf(t) - l(t) - \lambda(f(t) - \frac{x}{\beta})]dt \tag{41}$$

where

$$f(t) = f(l(t) - b(t), t)$$

and

$$\beta = \text{length of } T$$

From this we obtain the following Euler condition for optimality:

$$(P - \lambda)\frac{\partial f(t)}{\partial l} = 1 \tag{42}$$

for all t. This implies that the marginal products of labor are equal for all firms within the industry.

Next we vary x and $\alpha = t_s$ (a right end point of a sub-interval). Assuming that the solutions, still denoted $l(\cdot), x(\cdot)$ etc., are unique and differentiable with respect to x and α , we have

$$\int_T [P'f(t) - ((P - \lambda)f'(t) - 1)\frac{\partial l}{\partial x} + \frac{\lambda}{\beta}]dt = 0 \tag{43}$$

and,

$$\begin{aligned} Pf(\alpha) &- l(\alpha) - \lambda(\alpha)f(\alpha) + \frac{\lambda x}{\beta} \\ &+ \int_T [((P - \lambda)f'(t) - 1)\frac{\partial l(\cdot)}{\partial \alpha} - \frac{\lambda x}{\beta^2}]dt = 0 \end{aligned} \tag{44}$$

where we have set

$$P' = \frac{dP}{dx} \text{ and } f'(t) = \frac{\partial f(t)}{\partial l(t)}.$$

In view of (42) and (43), we then have

$$\begin{aligned} -\int_T P'f(t)dt &= \lambda \\ &= P - \frac{1}{f'(t)} \quad (t \in T) \end{aligned} \tag{45}$$

We note that $\lambda > 0$ since $P' > 0$. Hence, noticing that P' is independent of t and using (40), we have

$$P(x) + xP'(x) = \frac{1}{f'(t)}. \tag{46}$$

On the other hand, (42) and (44) yield

$$\frac{f(\alpha)}{l(\alpha)} = f'(\alpha) \tag{47}$$

for each $\alpha = t_s$ $(s = 1, 2, ..., s_i)$. Since $1/f'(t)$ is the marginal cost (MC) of the product equation (46) may be expressed as

$$-\frac{x}{p} \cdot \frac{dP}{dx} = (p - MC)/p. \tag{48}$$

Combining (42), (46) and (47) we can state

Lemma 1.

Under assumptions (A.1)-(A.5), the profit of each industry in Model D is maximized if (i) marginal products of labor are equal for all firms, (ii) the mark up ratio equals the elasticity of inverse demand function for the product and (iii) the marginal cost equals the average cost of the marginal firm.

Next we consider a slightly different problem. Suppose that firms in the industry maximize their joint profit, π as in the previous analysis, but T is now under the control of government. T will be chosen so that the utility of the representative consumer is maximized given the behavior of the monopolistically competitive firms.

For each $T \in \underline{T}$ let $\tilde{l}(\cdot), \tilde{x}(\cdot)$, and $\tilde{x}$ be the solutions of the problem (P) (hence these solutions satisfy (42),(46) and (47)). In the sequel, the tilde sign over the functions will be deleted. We consider the following problem:

(P) Maximize

$$\begin{aligned} U &= u(x) + a - l \\ &= u\left(\int_T f(t)dt, a - \int_T l(t)dt\right) \end{aligned} \tag{49}$$

with respect to T where $l(t)$ is the solutions of the problem stated above.

Consider a change in $\alpha = t_s$, one of the right end points of the sub-intervals in (A.5). Differentiating (49), along the optimal solution, with respect to α, we have

$$\begin{aligned} \frac{dU}{d\alpha} &= U'(x)f(\alpha) + \int_T f'(t)\frac{dl}{d\alpha}dt \\ &+ \ l(\alpha) + \int\left(\frac{\partial l}{\partial \alpha}dt\right) \end{aligned} \tag{50}$$

If the consumer maximizes utility at given market prices, then

$$\frac{du}{dx} = p, \tag{51}$$

hence if we define

$$s = (p - MC)/MC, \tag{52}$$

where MC is the marginal cost of the industry, we find from (46)(noticing that $MC = 1/f'$), that s is positive. Also (50) may be expressed as

$$\frac{dU}{d\alpha} = pf(\alpha) - l(\alpha) - s\int_T \frac{dl}{d\alpha}. \tag{53}$$

Now differentiating (46) with respect to α and noticing that $f'(t)$ is independent of t we have

$$\begin{aligned}(2P'(x) \quad &+ \quad xP"(x))(f'(t)\int_T \frac{dl(t)}{d\alpha} + f(\alpha)) \\ &= \quad -\frac{f"(t)}{(f'(t))^2} \cdot \frac{dl(t)}{d\alpha}\end{aligned} \tag{54}$$

Finally we assume that

Assumption C.2

All incumbent firms either decrease or increase labor inputs if there is an entry of a marginal firm.

In view of (54) and assumptions on the sign of derivatives of functions we can show that $dl(t)/d\alpha > 0$. We have thus proved (see, (54):

Theorem 3

The marginal contribution of a firm of in model D is equal to the difference between (i) the profit of the firm and (ii) the increase in the total costs of all monopolists each multiplied by the corresponding mark up ratio, s. This second term takes on a positive value.

Remarks (a) Notice that if we denote the demand elasticity of the good (the reciprocal of the left side of (46)) by e , we have

$$s = \frac{1}{e-1}. \tag{55}$$

Hence (53) is in accordance with the formula of Kahn [(1962) p.29], which was obtained in a partial equilibrium framework. Notice that although he did not assume the joint profit maximization, he did assume that the mark up ratio is constant for all firms in the industry. As to the simplifying assumption on which this result depends see McKenzie(1951).

(b) As an alternative interpretation of the present model, assume that the industry is monopolized by a firm which has a continuum of potential factories, T. Then the maximization of the profit of the monopolist can be analyzed in exactly the same way as in the present model.

In the last interpretation, in view of (51), we have the following result:

Thorem 4a

In the monopolistic market, if the firm operates its factories until the last of them earn zero profit, the contribution is positive. Hence entry is excessive.

This corresponds to the content of the excess entry theorem in Suzumura andKiyono (1987), which was established for the homogeneous good Cournot-type oligopoly model. Weizsäcker(1980) analyses a heterogeneous duopoly model with a quadratic utility function.

(c) The We above framework may be interpreted as a model of monopolistic competition, as formulated by Chamberlin(1933), with a large (non-oligopolistic) group of suppliers of physically similar but economically differentiated products. Bishop(1976) analyzed the welfare implication of equilibrium of the market where, as in the Chamberlin's idealization, all the actual and potential members of the group have the "same" costs and face the "same" demands. He showed diagramatically that, in the monopolistic competition market, entry is excessive from the consumer's viewpoint. The proposition was generalized in the present analysis to the case where the production function (cost functions) of firms in the industry may be different.

Theorem 4b

In the monopolistic competition model, where all the actual and potential firms in the industry face the same demands, the optimal product variety calls for production at a point short of minimum average cost of the marginal firm.

This result is a direct consequence of (53). We need to interpret that the domain of imtegration $\underline{T_i}$ now represents the variety of the (physically identical) products in the industry.

5 Appendix to Section 3:The Existence of the Optimum and Related Topics

In this section we will prove the existence of solutions to problem (P_T) in models A and B, and discuss the continuity, with respect to t $(t \in T)$, of the solutions, in each of the fixed sub-intervals in the domain of integration. Detailed proofs will be given only for model B, because the proofs for model A are essentially the same and even simpler.

Model D

This is a modification of model B, with many of the technical assumptions generalized. Let n be the number of goods in the economy. To simplify the argument we assume that good n is labor. For $i = 1, 2, ..., n-1$, let $(\underline{T_i}, B_i, \mu)$ be a measure space where $\underline{T_i}$ is a bounded interval in the real line R, B_i, is the σ-algebra of Borel sets of $\underline{T_i}$ and μ is the Lebesque measure. As before $\underline{T_i}$ is the set of potential firms in industry i, and each member T_i of B_i is interpreted as the set of firms that are actually producing positive outputs in industry i. Problem (P_T) is formulated as in model B. But this time we choose T_n to be empty (labor is never produced). Hence the problem reduces to:

(P_T) Maximize

$$c_n = -\int_T \sum_{j \neq n} y_n^j(t) \tag{56}$$

subject to

$$c_i = \int_T (f^i(\tilde{y}^i(t) - \tilde{b}^i(t), t) - \sum_{j \neq n} y_n^j(t))dt \quad (i = 1, 2, ..., n-1). \tag{57}$$

Since f^i is assumed to be increasing, (P_T) is unaltered if we replace the inequalities by equalities. Instead of (B1) we make the following:

Assumption D.1

For each $i \in N$, (i) $f^i(\tilde{y}^i(t), t)$ is continuous for almost all $(\tilde{y}^i(t), t) \in R_{n-1}^+ \times T_i$ (R_{n-1}^+ denotes the non-negative orthant of $n-1$ dimensional Euclidean space) and (ii) $f^i(\tilde{y}^i(t), t)$ is increasing in $\tilde{y}^i(t)$, for almost all $t \in T_i$.

For some of the arguments below it is enough to replace (i) by (i)' for almost all $t, f^i(\cdot, t)$ is upper semi-continuous and, for almost all $\tilde{y}^i(t)$ $f^i(\tilde{y}^i(t), \cdot)$ is measurable. Such a numerical function is usually referred to as a *Carathédory* function (in a minimization problem $f^i(\cdot, t)$ is assumed to be lower semi-continuous). It is a special case of a normal integrand (see, e.g., Ekeland-Temam (1976) pp.231-234]), all of which satisfy the condition that (i)" for almost all $t, f^i(\cdot,t)$ is upper semi-continuous and there exists a Borel function $\tilde{f}^i$ such that $\tilde{f}^i(\tilde{y}^i,\cdot) = f^i(\tilde{y}^i,\cdot)$ for almost all $\tilde{y}^i$.

All functions that we consider are assumed to be integrable. We now make

Assumption D.2

For each $i = 1,2,...,n-1, c_i > 0$, and it is technologically possible to satisfy net demand $c_i + d_i (i = 1,2,...,n-1)$ for some $d_i > 0$ (i.e., (57) has solutions $\tilde{y}^i(t) \geq 0$ when each c_i is replaced by $c_i + d_i$)

The assumption on the sign of c_i's is made mainly for simplicity of exposition. It is very easy to cover the case where some of them are negative (the case of primary factors of production).

In order to rule out the possibility that the production of a good will be carried out by a negligibly small set of firms, we need a certain uniformity assumption on the production technology. To simplify the argument we assume that, given the set of active firms in an industry and the net final demand for the good, there are lower bounds such that if the members of a non-negligible set of firms are using inputs beyond any of the bounds, then there exists a more efficient way of allocating resources within each industry. More precisely, we make

Assumption D.3 (inefficiency of over concentration)

For each $(i = 1,2,...,n-1)$ there exists $\tilde{a}^i \in R_+^{n-1}$ (which may depend on c_i and T_i) such that if not $\tilde{y}^i(t) \leq \tilde{a}^i$ for almost all t in some non-null set $S_i \subset T_i$, there exists $\hat{y}(t) \in R_+^{n-1}$ such that $\hat{y}(t) \leq \tilde{a}^i$ for all T_i,

$$\int_{Ti} \hat{y}^i(t)dt \leq \int_{Ti} \tilde{y}^i(t)dt$$

and

$$\int_{Ti} f^i(\tilde{y}^i(t)dt \leq \int_{Ti} f^i(\hat{y}^i(t),t)dt$$

This assumption is likely to be satisfied if firms in an industry can be classified into a finite number of groups with positive measures, in such a way that firms within each group are technologically "similar" and there are "no increasing returns" in production. Because of this assumption we may suppose that the optimal solution of (P_T) lies in a compact set defined by $\tilde{a}^i$ $(i = 1,2,...,n-1)$.

Finally we will give a simple definition. Let f and g be functions from $X \times T \text{to} \bar{R}$ (the extended real line). We say that f is integrably dominated by g if, for every $\varepsilon > 0$, there exists a positive integrable function, $e(t)$, such that

$$f(x,t) \geq e(t) \text{ implies } f(x,t) \leq \varepsilon\ g(x,t)$$

We are now ready to state the following theorem

Theorem 5

Under assumptions D.1, D.2, D.3, A.3, and A.5 there exists a solution to problem (P_T) in model D, where the admissible solutions are taken to be all measurable functions..

The proof of Theorem 5 depends heavily on the following proposition due to Berliocchi and Lasry ((1973) pp. 155-156), which is an extension of the main theorem of Aumann and Perles (1965).

Theorem A.

Let $g^n : R^{n-1} \times T \to R$ be a Borel function such that $x \to g^n(x,t)$ is upper semi-continuous almost everywhere and $g^1, g^2, ..., g^{n-1}$ be normal integrands of $R^{n-1} \times R_+ \to R$. If (α) sup $(0, g^n)$ is integrably dominated by $g^1 + g^2 + ... + g^{n-1}$ and (b) lim $(g^1 + g^2 + ... + g^{n-1})(x,t) \to \infty$ as $\|x\| \to \infty$ almost everywhere, then the problem

(Q) maximize

$$\int_T g^n(x(t), t)dt$$

subject to

$$\int_T g^i(x(t), t)dt \ \leq \ k_i \qquad (i = 1, 2, ..., n-1)$$

(where $k_i > 0$) has a solution. If the domain of g^n is $S \times T$, where S is compact, then the assumption on the asymptotic behavior of $\sum g^i$ can be dropped.

(Proof of Theorem 5) We define

$$x(t) = (\tilde{y}^1(t), \tilde{y}^2(t), ..., \tilde{y}^{n-1}(t), 0) \in R_+^{(n-1)n} \tag{58}$$

$$g^n(x,t) = -\sum_{j \neq n} \tilde{y}_n^j \tag{59}$$

Also using Assumption (D.2) we may add a positive number to each of equalities in (56) and (57) and assume that the right hand side of each of them are non-negative. This proves the existence of a solution to (P_T).

The existence of a solution to problem (P_T) in model A can be proved in a very similar way. The key to the proof is the following proposition of Berliocchi and Lasry ((1973) p.155).

Theorem B.

Let $f^i : X \times T \to \bar{R}$ $(i = 1, 2, ..., k)$ be Carathédory functions and $g^i : X \times T \to \bar{R}$ $(i = 1, 2, ..., n)$ be normal integrands. If lim $\sum g^i \to \infty$ almost everywhere and each $|f^i|$ is integrably dominated by $\sum g^i$ and $u : R^k \to R$ is continuous, then problem

(Q) maximize

$$u(\int_T f^1(x(t),t)dt, ..., \int_T f^k(x(t),t)dt)),$$

subject to

$$\int_{Ti} g^i(x(t),t)dt \leq 1 \qquad (i = 1, 2, ..., n)$$

has an optimal solution.

In Theorem 5 we gave conditions under which there exists a measurable solution, $\tilde{y}^i(t)$ $(i \in N)$, to the problem (P_T). Let us next give conditions under which these functions are chosen to be continuous in each of the subintervals of T_i. For each $\delta_i, 0 < \delta_i < d_i$, where d_i is defined in (D.2), we consider a "perturbed problem" :

(P_δ) Minimize

$$\int_T \sum_{j \neq n} y_n^i(t)dt \tag{60}$$

subject to

$$-(c_i + \delta_i) \geq \int_T (\sum_{j \neq i} y_i^j(t) - f^i(\tilde{y}^i(t) - \tilde{b}^i(t), t)dt \qquad (i = 1, 2, ..., n-1). \tag{61}$$

We set

$$\delta = (\delta_1, ..., \delta_{n-1})$$

and

$$h(\delta) = \inf(P_\delta), \tag{62}$$

namely, the infimum of problem P_δ

We also set

$$L(x, t, \delta^*) = g^n(x, t) + \sum_{j=1}^{n-1} \delta_j^* g^j(x, t) \tag{63}$$

where we define x by (58) and $g^i(x,t)(i \in N)$ by

$$g^n(x, t) = \sum_{j \neq n} y_n^j \tag{64}$$

and

$$g^i(x, t) = \sum_{j \neq i} y_i^j - f^i(\tilde{y}^i - \tilde{b}^i, t) \quad (i = 1, 2, ..., n-1) \tag{65}$$

We make

Assumption D.4

For every non-negative and non-zero $\delta^* \in R^{n-1}$ and almost all $t \in T_i$, there exists single $x \in R^{(n-1)n}$ such that $L(x, t, \delta^*)$ is a minimum.

We notice that this assumption is satisfied if, for example, all functions $f^i(\tilde{y}^i(t), t)$ are strictly concave in $\tilde{y}^i(t)$ (since then L in (63) is strictly convex in x).

We will show

Proposition 1

Under assumptions D.1 D.4, A.3, and A.5, problem (P_T) in model D has a solution which is continuous in t in each sub-interval defined in A.3.

The following proof depends heavily on the analysis in Ekeland and Temam ((1976) pp.367-373). We write the Lagrangean of (P_δ) as

$$\int_T L(x(t), t, \delta^*)dt \tag{66}$$

where L is defined by (63). By (D.3) we may assume that $x(t)$ lies in a compact set K. Hence applying the measurable selection theorem (Ekerland and Temam (1976) p.236), we can find a measurable function $\gamma(t, \delta^*)$ such that

$$\gamma(t, \delta^*) = \min\{L(x, t, \delta^*)/x \in K\} \tag{67}$$

and

$$\min \int_T L(x(t), t, \delta^*)dt = \int_T \gamma(t, \delta^*)dt. \tag{68}$$

We define $\bar{x}(t)$ by

$$L(\bar{x}(t), t, \delta^*) = \gamma(t, \delta^*). \tag{69}$$

It can be shown ((1976) pp.367-373) that δ^* is a sub-gradient of $h(\delta)$, which is non-empty in the neighborhood of zero because of (D.2).For fixed δ (in particular for $\delta = 0, L(x, t, \delta^*)$ is continuous in x and t. Hence, by the maximum theorem (see, e.g., Corollary to Theorem 3 in B of Hildenbrand (1974)), $\bar{x}(t)$ is a non-empty and upper hemi-continuous set-valued mapping. Our uniqueness assumption (D.4) then implies that $\bar{x}(t)$ (and hence each $\tilde{y}^i(t)$) is a continuous function.

Under the assumptions of the previous theorem, $\bar{x}(t)$ is continuous in each of the sub-intervals of T_i. If these intervals are taken to be compact, $\bar{x}(t)$ is a function of bounded variation(Dunford-Schwartz (1958)), and hence is differentiable with respect to t almost everywhere in the sub-intervals. This implies that under the assumptions of the theorem we may assume that all solution functions, $\tilde{y}^i(t))$, are differentiable almost everywhere. We are thus in the realm of the ordinary theory of the calculus of variations, and so the assumptions we made in model B are justified.

References

[1] Arrow, K. J. and F. H. Hahn (1971): *General Competitive Analysis*, San Francisco: Holden-Day.

[2] Aumann, R. J (1964): "Markets with a Continuum of Traders," *Econometrica* Vol. 32 pp.39-50.

[3] ————-(1975): "Values of Markets with a Continuum of Traders," *Econometrica* Vol. 43 pp.611-647.

[4] Aumann, R. J. and M. Perles (1965): "A Variational Problem Arising in Economics," *Journal of Mathematical Analysis and Applications*, 11 pp.488-503.

[5] Aumann, R. J. and L. Shapley (1971): *Values of Non-Atomic Games*, Princeton, N. J: Princeton University Press.

[6] Berliocchi, H. and J. M. Lasry (1973): "Integrandes normales et measures parametrees en calcul des variations," *Bulletin de la Societe Mathematique de France*, Vol. 101 pp.129-184.

[7] Bishop,R. (1967): "Monopolistic Competition and Welfare Economics", in R.E.Kuenne ed. *Monopolistic Competition Theory :Studies in Impact* pp.251-263.

[8] Chamberlin E.H. (1933): *The Theory of Monopolostic Competition*, Harvard University Press

[9] Debreu, G (1959): *Theory of Value* New York; John Wiley and Sons.

[10] Dunford, N. and J. Schwartz (1958): *Linear Operators I*, New York; Interscience.

[11] Ekeland, I. and R. Temam (1976): *Convex Analysis and Variational Problems*, Amsterdam; North-Holland.

[12] Gelfand, I. M. and S. V. Fomin (1963): *Calculus of Variations*, Englewood Cliffs, N.J.; Prentice-Hall.

[13] Hicks, J. R (1939): "Foundation of Welfare Economics", *Economic Journal*, Vol. 49 pp.696-711.

[14] Hildenbrand, W. (1974): *Core and Equilibria of A Large Economy*, Princeton, N.J.; Princeton University Press.

[15] Ichiishi, T. (1981): "Super-modularity: Applications to Convex Games and to the Greedy Algorithm for LP", *Journal of Economic Theory*. Vol. 25 pp.283-286.

[16] Kahn, R. F. (1935): "Some Notes on Ideal Output", *Economic Journal*, Vol. 45 pp. 1-35.

[17] Makowski, L. (1980): "Perfect Competition, the Profit Criterion, and the Organization of Economic Activity", *Journal of Economic Theory*, Vol.22 pp.222-242.

[18] Meade, J.E. (1937): *Economic Analysis and Policy*, Oxford University Press.

[19] McKenzie, L.W. (1951): "Ideal Putput and the Interdependence of Firms", *Economic Journal*. Vol. 61 pp.785-803.

[20] Negishi, T. (1962): "Entry and Optimal Number of Firms", *Metroeconomica*, Vol. 14 pp.86-96.

[21] ————-(1972): *General Equilibrium Theory and International Trade*, Amsterdam: North-Holland.

[22] Ostroy,J. M. (1980): "The No-Surplus Condition as a Characterization of Perfectly Competitive Equilibrium", *Journal of Economic Theory*, Vol.22 pp.183-207.

[23] Samuelson, P. A.(1947): *Foundations of Economic Analysis*, Cambridge, Mass: Harvard University Press.

[24] Shapley, L.S. (1953): "A Value for n-Person Games," in *Contribution to the Theory of Games Vol.II*, ed. by H. W. Kuhn and A. W. Tucker, Princeton, N.J.: Princeton University Press.

[25] ————(1971): "Cores of Convex Games", *International Journal of Game Theory*. Vol.1 pp.11-26.

[26] Suzumura, K. and K. Kiyono (1987): "Entry Barriers and Economic Welfare", *Review of Economic Studies*. Vol.54 pp.157-167.

[27] Topkis, D.M. (1987): "Activity Optimization Games with Complementarity", *European Journal of Operations Research*, Vol.28 pp358-368.

[28] von Weizsäcker,C.C. (1980). "A Welfare Analysis of Barriers to Entry", *Bell Journal of Economics* , Vol.12 pp.399-420.